Differential Geometry III: Riemannian Geometry

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9.1 Let $(\mathcal{M}, g)$ be an n-dimensional Riemannian manifold. Recall that, in any local coordinate system, the Ricci tensor satisfies

$$Ric_{ij} = g^{\alpha\beta} R_{\alpha i \beta j}.$$

(a) Show that the Ricci curvature is symmetric, i.e. for any $X, Y \in \Gamma(\mathcal{M})$:

$$Ric(X, Y) = Ric(Y, X).$$

(b) The symmetries of the Riemann curvature tensor imply that not all components R_{ijkl} of the Riemann tensor are independent of each other. How many independent components does R have when $n = 2$? Show that, in this case, for any $X, Y, Z, W \in \Gamma(\mathcal{M})$

$$R(X, Y, Z, W) = K \cdot \big(g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\big),$$

where K is the sectional curvature of $\mathcal{M}$ (since $\dim\mathcal{M} = 2$, there is only one tangent 2-plane passing through each point $p \in \mathcal{M}$; hence, in this case, the sectional curvature is simply a function on $\mathcal{M}$).

(c) How many independent components does R have when $n = 3$? Show that, in this case,

$$R_{ijkl} = \mathrm{Ric}_{ik}\, g_{jl} - \mathrm{Ric}_{il}\, g_{jk} + \mathrm{Ric}_{jl}\, g_{ik} - \mathrm{Ric}_{jk}\, g_{il} - \frac{1}{2} S (g_{ik} g_{jl} - g_{jk} g_{il}),$$

where $S = g^{ij}\mathrm{Ric}_{ij}$ is the scalar curvature; in particular, the Ricci curvature contains, in this case, all the information about the Riemann curvature tensor.

Solution. (a) It suffices to show that, for any $p \in \mathcal{M}$ and in any local coordinate system $(x^1, \ldots, x^n)$ around p, we have

$$Ric_{ij} = Ric_{ji}$$

(since $Ric(X, Y) = Ric_{ij} X^i Y^j$). This follows directly from the definition of the Ricci curvature tensor

$$Ric_{ij} = g^{ab} R_{aibj},$$

together with the fact that $g^{ab} = g^{ba}$ (since the matrix of components of g is symmetric) and the identity $R_{aibj} = R_{bjai}$.

(b) In any dimension, the components of the Riemann curvature tensor in any local coordinate system satisfy the following symmetries:

$$R_{abcd} = -R_{bacd} = -R_{abdc}, \qquad (1)$$
$$R_{abcd} = R_{cdab}.$$

In the special case when $n = 2$, the indices take the values $\{1, 2\}$. In view of the first symmetry above, we know that any component R_{abcd} with $a = b$ or $c = d$ has to vanish identically. Therefore, the only components of R that can be non-zero have to include two indices equal to 1 and the other two equal to 2 (and the first two or the last two indices cannot be the same). Combining the two symmetries

(1), we infer that the only independent component of R is R_{1212}, since all other components of R are either identically 0 or are equal to $\pm R_{1212}$ (obtained by a sequence of permutations of the indices of R_{1212} of the type that appear in (1)).

Let $p \in \mathcal{M}$. We want to show that, for any $X, Y, Z, W \in T_p\mathcal{M}$,

$$R(X,Y,Z,W)|_p = K|_p \cdot \big(g|_p(X,Z)g|_p(Y,W) - g|_p(X,W)g|_p(Y,Z)\big). \tag{2}$$

Since this is a geometric relation, it suffices to show that it is true in a frame $\{e_1, e_2\}$ of $T_p\mathcal{M}$. To this end, let us pick $\{e_1, e_2\}$ to be an orthonormal frame of $(T_p\mathcal{M}, g|_P)$ (so that $g|_p(e_i, e_j) = \delta_{ij}$). Recall that, for such a frame, the sectional cruvature $K|_p$ satisfies

$$K|_p = R(e_1, e_2, e_1, e_2).$$

Therefore, we have

$$R(e_1,e_2,e_1,e_2) = K|_p \cdot \big(g|_p(e_1,e_1)g|_p(e_2,e_2) - g|_p(e_1,e_2)g|_p(e_1,e_2)\big),$$

i.e. (2) is true for $X = Z = e_1$, $Y = W = e_2$. From this, we can directly infer that (2) is true when $X, Y, Z, W \in \{e_1, e_2\}$ using the fact that the right hand side of (2) satisfies the same symmetries (1) as the left hand side with respect to permutations of the arguments. Finally, (2) for any $X, Y, Z, W \in T_p\mathcal{M}$ follows by expressing each of those vectors in the basis $\{e_1, e_2\}$ and using the multilinearity of (2) in all of its arguments.

(c) In the case when $n = 3$, we want to show that R_{abcd} has at most 6 independent compoents (as many as the Ricci tensor, whose components form a 3×3 symmetric matrix). Using the fact that R_{abcd} satisfies the symmetries (1), we deduce that, for a non-zero component R_{abcd}, no three indices can be the same number (and we cannot have $a = b$ or $c = d$). Since a, b, c, d take the values $\{1, 2, 3\}$, this observation allows us to form a list of components R_{abcd} from which all non-zero components can be computed via the set of symmetries (1):

1. Components with two indices equal to “1”: R_{1212}, R_{1213}, R_{1313} (all other non-zero components in this category can be obtained from one of these three via a permutation of the indices of the type appearing in the symmetries (1)).

2. Components with one index equal to “1”: R_{1232}, R_{1323} (as before, all other non-zero components with one index equal to “1” can be obtained from these two via a permutation of the indices of the type appearing in the symmetries (1)).

3. Components with no index equal to “1”: R_{2323} (same as before; this case is essentially the same as for $n = 2$).

All in all, we infer that every non-zero component R_{abcd} is equal to ± 1 times one of the 6 components R_{1212}, R_{1213}, R_{1313}, R_{1232}, R_{1323} and R_{2323}. Since the symmetries of the Riemann tensor only refer to permutation of the indices and none of these components is simply an index-permutation of another one, we infer that R_{abcd} in the case $n = 3$ has precisely 6 independent components.

We will now show how the Riemann tensor can be computed in terms of the Ricci tensor when $n = 3$. We want to show that, in any local coordinate system,

$$R_{ijkl} = \mathrm{Ric}_{ik}\, g_{jl} - \mathrm{Ric}_{il}\, g_{jk} + \mathrm{Ric}_{jl}\, g_{ik} - \mathrm{Ric}_{jk}\, g_{il} - \frac{1}{2}S(g_{ik}g_{jl} - g_{jk}g_{il}). \tag{3}$$

Even though the above relation looks coordinate dependent, it is not; you can easily check that it is equivalent to the coordinate-free statement that, for any $p \in \mathcal{M}$ and any $X, Y, Z, W \in T_p\mathcal{M}$,

$$\begin{aligned} R(X,Y,Z,W) = \mathrm{Ric}(X,Z)\, g(Y,W) - \mathrm{Ric}(X,W)\, g(Y,Z) + \mathrm{Ric}(Y,W)\, g(X,Z) - \mathrm{Ric}(Y,Z)\, g(X,W) \\ - \frac{1}{2}S\big(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\big). \end{aligned}$$

Therefore, for any $p \in \mathcal{M}$, it suffices to establish (3) at p with respect to a single well-chosen coordinate system around p. To this end, let us choose a normal coordinate system (x^1, x^2, x^3) around p (so that $g_{ij}|_p = \delta_{ij}$ and $\partial_i g_{jk}|_p = 0$). In these coordinates, we have seen in class that the components of the Riemann curvature tensor $R|_p$ at p take the simple form

$$R_{abcd}|_p = \frac{1}{2}\big(\partial_b\partial_c g_{ad}|_p - \partial_a\partial_c g_{bd}|_p + \partial_a\partial_d g_{bc}|_p - \partial_b\partial_d g_{ac}|_p\big).$$

Therefore,

$$\begin{aligned} Ric_{ab}|_p &\doteq g^{ij}|_p R_{aibj}|_p = \delta^{ij} R_{aibj}|_p \\ &= \frac{1}{2}\sum_{i=1}^{3}\Big(\partial_i\partial_b g_{ia}|_p + \partial_i\partial_a g_{ib}|_p - \partial_a\partial_b g_{ii}|_p - \partial_i^2 g_{ab}|_p\Big) \end{aligned}$$

and

$$S|_p \doteq g^{ij}|_p Ric_{ij}|_p = \sum_{i,j=1}^{3}\Big(\partial_i\partial_j g_{ij}|_p - \partial_i^2 g_{jj}|_p\Big).$$

Using the above expressions, verifying the relation

$$\begin{aligned} R_{abcd}|_p &= \big(\mathrm{Ric}_{ac}\, g_{bd} - \mathrm{Ric}_{ad}\, g_{bc} + \mathrm{Ric}_{bd}\, g_{ac} - \mathrm{Ric}_{bc}\, g_{ad} - \frac{1}{2}S(g_{ac}g_{bd} - g_{bc}g_{ad})\big)|_p \\ &= \mathrm{Ric}_{ac}|_p\, \delta_{bd} - \mathrm{Ric}_{ad}|_p\, \delta_{bc} + \mathrm{Ric}_{bd}|_p\, \delta_{ac} - \mathrm{Ric}_{bc}|_p\, \delta_{ad} - \frac{1}{2}S|_p(\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad}) \end{aligned}$$

is a straightforward (yet somewhat tedious) algebraic task (one only needs to verify this for the 6 independent components R_{abcd} identified earlier).

9.2 Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold and let $\phi : (-\epsilon, \epsilon) \times [0,1] \to \mathcal{M}$ be a smooth map such that, for each $s \in (-\epsilon, \epsilon)$, $\gamma_s = \phi(s, \cdot)$ is a *geodesic*. Define the vector fields $T = d\phi(\frac{\partial}{\partial t})$ and $X = d\phi(\frac{\partial}{\partial s})$. Prove that

$$\nabla_T\nabla_T X = -R(X,T)T.$$

Intuitively, X measures the infinitesimal separation between nearby geodesics; thus, the Riemann curvature tensor "measures" the relative acceleration of nearby geodesics (compare the behaviour of nearby geodesics in the Euclidean plane vs. the round sphere).

Solution. Since γ_s is assumed to be a geodesic for each s, the vector field $T \doteq \phi_* \frac{\partial}{\partial t} = \dot{\gamma}_s$ satisfies

$$\nabla_T T = 0.$$

Moreover, as we have shown in class, the variation vector field X and the tangent vector field T commute with each other, since

$$[X,T] = [\phi_* \frac{\partial}{\partial s}, \phi_* \frac{\partial}{\partial t}] = \phi_* [\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0.$$

Therefore, the definition of the Riemann curvature tensor and the fact that ∇ is torsion-free imply that:

$$\begin{aligned} R(X,T)T &= \nabla_X \nabla_T T - \nabla_T \nabla_X T - \nabla_{[X,T]} T \\ &= 0 - \nabla_T \nabla_X T - 0 \\ &= -\nabla_T \big(\nabla_T X + [X,T] \big) \\ &= -\nabla_T \nabla_T X, \end{aligned}$$

i.e. we obtain the desired relation.

9.3 Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold. For any smooth curve $\gamma : [0,1] \to \mathcal{M}$ and any $t_1, t_2 \in [0,1]$, we will denote with $\mathbb{P}_{\gamma(t_1) \to \gamma(t_2)} : T_{\gamma(t_1)} \mathcal{M} \to T_{\gamma(t_2)} \mathcal{M}$ the parallel transport along γ from $\gamma(t_1)$ to $\gamma(t_2)$ (with respect to the Levi-Civita connection).

(a) Prove that, for any vector field Z along γ, as $\tau \to 0$:

$$\lim_{\tau \to 0} \frac{Z|_{t=0} - \mathbb{P}_{\gamma(\tau) \to \gamma(0)} Z|_{t=\tau}}{\tau} = -\nabla_{\dot{\gamma}(0)} Z.$$

Hint: Contruct a frame $\{e_i\}_{i=1}^n$ *of vector fields along* γ *which are parallel translated, and express* Z *in components with respect to* e_i.

*(b) Let $\phi : [-1,1] \times [-1,1] \to \mathcal{M}$ be a smooth map with $p = \phi(0,0)$ and let $X = \phi^*(\frac{\partial}{\partial x^1})$ and $Y = \phi^*(\frac{\partial}{\partial x^2})$. For any $s_1, s_2 \in (0,1)$, we will consider the rectangular loop $\gamma_{(s_1,s_2)}$ starting and ending at p which is of the form $\gamma_{(s_1,s_2)} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$\begin{aligned} \gamma_1(t) &= \phi(t, 0), \quad t \in [0, s_1], \\ \gamma_2(s) &= \phi(s_1, s), \quad s \in [0, s_2], \\ \gamma_3(t) &= \phi(s_1 - t, s_2), \quad t \in [0, s_1], \\ \gamma_3(s) &= \phi(0, s_2 - s), \quad s \in [0, s_2]. \end{aligned}$$

For any $Z \in T_p\mathcal{M}$, let $Z_{(s_1,s_2)} \in T_p\mathcal{M}$ be the tangent vector obtained after parallel transporting Z_p around γ, i.e. following the successive mappings

$$\begin{aligned} Z \to & Z' = \mathbb{P}_{\gamma_1(0) \to \gamma_1(s_1)} Z \to Z'' = \mathbb{P}_{\gamma_2(0) \to \gamma_2(s_2)} Z' \\ & \to Z''' = \mathbb{P}_{\gamma_3(0) \to \gamma_3(s_1)} Z'' \to Z_{(s_1,s_2)} = \mathbb{P}_{\gamma_4(0) \to \gamma_4(s_2)} Z'''. \end{aligned}$$

Show that

$$\lim_{s_2\to 0}\lim_{s_1\to 0}\frac{Z_{(s_1,s_2)}-Z}{s_1 s_2}=-R(X,Y)Z.$$

Thus, the Riemann curvature tensor quantifies the failure of the parallel transport around small closed loops to be the identity map.

Solution. (a) Let $\{\xi_\alpha\}_{\alpha=1}^n$ be a basis of orthonormal tangent vectors in $T_{\gamma(0)}\mathcal{M}$ with respect to $g|_{\gamma(0)}$ and let $\{e_\alpha\}_{\alpha=1}^n$ be a set of vector fields along γ such that e_α is the parallel translate of ξ_α (i.e. $e_\alpha|_{t=0}=e_\alpha$ and $\nabla_{\dot\gamma}e_\alpha=0$. Since

$$\frac{d}{dt}g(e_\alpha,e_\beta)|_{\gamma(t)}=g(\nabla_{\dot\gamma}e_\alpha,e_\beta)+g(e_\alpha,\nabla_{\dot\gamma}e_\beta)=0,$$

we infer that, for any $t\in[0,1]$, $\{e_\alpha|_{\gamma(t)}\}_{\alpha=0}^n$ is an orthonormal base for $T_{\gamma(t)}\mathcal{M}$.

Any vector field Z along γ can be expressed, with respect to the basis $\{e_\alpha\}_{\alpha=0}^n$ as $Z=Z^\alpha e_\alpha$ for some (unique) component functions $Z_\alpha:[0,1]\to\mathbb{R}$, $\alpha=0,\dots,n$. In this basis, the covariant derivative and the parallel translation of a vector field become a standard derivative and translation, respectively, of the component functions; in particular, we can readily compute:

$$\nabla_{\dot\gamma}Z=\nabla_{\dot\gamma}(Z^\alpha e_\alpha)=\frac{dZ^\alpha}{dt}e_\alpha+Z^\alpha\nabla_{\dot\gamma}e_\alpha=\frac{dZ^\alpha}{dt}e_\alpha.$$

Moreover, since, for any $t_1,t_2\in[0,1]$, we have $\mathbb{P}_{\gamma(t_1)\to\gamma(t_2)}e_\alpha|_{\gamma(t_1)}=e_\alpha|_{\gamma(t_2)}$, the linearity of the parallel transport operator implies that if $v=v^\alpha e_\alpha|_{\gamma(t_1)}$ is an element of $T_{\gamma(t_1)}\mathcal{M}$, then

$$\mathbb{P}_{\gamma(t_1)\to\gamma(t_2)}v=v^\alpha e_\alpha|_{\gamma(t_2)}.$$

We can thus calculate:

$$\begin{aligned}\lim_{\tau\to 0}\frac{Z|_{t=0}-\mathbb{P}_{\gamma(\tau)\to\gamma(0)}Z|_{t=\tau}}{\tau}&=\lim_{\tau\to 0}\frac{Z^\alpha(0)e_\alpha|_{t=0}-\mathbb{P}_{\gamma(\tau)\to\gamma(0)}(Z^\alpha(\tau)e_\alpha|_{t=\tau})}{\tau}\\&=\lim_{\tau\to 0}\frac{Z^\alpha(0)e_\alpha|_{t=0}-Z^\alpha(\tau)e_\alpha|_{t=0}}{\tau}\\&=-\frac{dZ^\alpha}{dt}(0)e_\alpha\\&=-\nabla_{\dot\gamma(0)}Z.\end{aligned}$$

Moreover, using Taylor's theorem to express for any $t\in[0,1]$:

$$Z^\alpha(t)=Z^\alpha(0)+\frac{dZ^\alpha}{dt}(0)t+\frac{1}{2}\frac{d^2Z^\alpha}{dt^2}(\xi(t))t^2$$

for some $\xi(t)\in[0,t]$ depending smoothly on t, we also have the following useful expression for the parallel transport operator:

$$\mathbb{P}_{\gamma(t)\to\gamma(0)}Z=Z|_{\gamma(0)}+\nabla_{\dot\gamma(0)}Z\cdot t+V[t]\cdot t^2 \tag{4}$$

for some smooth function $V : t \to V[t] \in T_{\gamma(0)}\mathcal{M}$ with $V^\alpha[t] = \frac{1}{2}\frac{d^2 Z^\alpha}{dt^2}(\xi(t))$; note that

$$V[t] \xrightarrow{t\to 0} \frac{1}{2}\nabla_{\dot\gamma}\nabla_{\dot\gamma} Z|_{t=0}.$$

(b) For $\phi : [-1,1]\times[-1,1] \to \mathcal{M}$ as in the statement of the exercise, we will denote by $\gamma_s(\cdot)$ the family of curves $t \to \phi(s,t)$ and by $\gamma'_t(\cdot)$ the family of curves $s \to \phi(s,t)$ in $\mathcal{M}$. Note that $X|_{\phi(s,t)} = \dot\gamma_s(t)$ and $Y|_{\phi(s,t)} = \dot\gamma'_t(s)$. For any vector field W defined along the image of the map ϕ and any $h \in (-1,1)$, we will define the vector fields $\mathbb{P}^{(h)}W$ on $\phi\big([-1,1]\times[-1+|h|, 1-|h|]\big)$ and $\mathbb{P}'^{(h)}W$ on $\phi\big([-1+|h|, 1-|h|]\times[-1,1]\big)$ to be the parallel translates of W along γ_s and γ'_t, respectively, with step h, i.e.

$$\big(\mathbb{P}^{(h)}W\big)|_{\phi(s,t)} = \mathbb{P}_{\gamma_s(t-h)\to\gamma_s(t)}W \quad \text{and} \quad \big(\mathbb{P}'^{(h)}W\big)|_{\phi(s,t)} = \mathbb{P}_{\gamma'_t(s-h)\to\gamma'_t(s)}W.$$

Note that, applying (4) for γ_s and γ'_t, we obtain

$$\begin{aligned}\big(\mathbb{P}^{(h)}W\big)|_{\phi(s,t)} &= W|_{\phi(s,t)} - (\nabla_X W)|_{\phi(s,t)}h + V_1[W;h]|_{\phi(s,t)}h^2\\ \big(\mathbb{P}'^{(h)}W\big)|_{\phi(s,t)} &= W|_{\phi(s,t)} - (\nabla_Y W)|_{\phi(s,t)}h + V_2[W;h]|_{\phi(s,t)}h^2\end{aligned}$$

for some smooth vector fields $V_1[W;h], V_2[W;h]$ depending smoothly on h and W and satisfying

$$V_1[W,h] \xrightarrow{h\to 0} \frac{1}{2}\nabla_X\nabla_X W, \quad V_2[W,h] \xrightarrow{h\to 0} \frac{1}{2}\nabla_Y\nabla_Y W. \tag{5}$$

Using the above formulas, we can compute for any $s_1, s_2 > 0$

$$\begin{aligned}\mathbb{P}'^{(s_2)}\mathbb{P}^{(s_1)}Z &= \mathbb{P}'^{(s_2)}\big(Z - (\nabla_X Z)s_1 + V_1[Z;s_1]s_1^2\big)\\ &= \big(Z - (\nabla_X Z)s_1 + V_1[Z;s_1]s_1^2\big) - \Big(\nabla_Y\big(Z - (\nabla_X Z)s_1 + V_1[Z;s_1]s_1^2\big)\Big)s_2\\ &\qquad + V_2\big[\big(Z - (\nabla_X Z)s_1 + V_1[Z;s_1]s_1^2\big); s_2\big]s_2^2\\ &= Z - (\nabla_X Z)s_1 - (\nabla_Y Z)s_2 + (\nabla_Y\nabla_X Z)s_1 s_2\\ &\qquad + V_1[Z;s_1]s_1^2 + V_2\big[\big(Z - (\nabla_X Z)s_1 + V_1[Z;s_1]s_1^2\big); s_2\big]s_2^2 - (\nabla_Y V_1[Z;s_1])s_1^2 s_2\end{aligned}$$

and, similarly,

$$\begin{aligned}\mathbb{P}^{(s_1)}\mathbb{P}'^{(s_2)}Z &= Z - (\nabla_X Z)s_1 - (\nabla_Y Z)s_2 + (\nabla_X\nabla_Y Z)s_1 s_2\\ &\qquad + V_2[Z;s_2]s_2^2 + V_1\big[\big(Z - (\nabla_Y Z)s_2 + V_2[Z;s_2]s_2^2\big); s_2\big]s_1^2 - (\nabla_X V_2[Z;s_2])s_2^2 s_1.\end{aligned}$$

Therefore, we compute

$$\begin{aligned}&\mathbb{P}'^{(s_2)}\mathbb{P}^{(s_1)}Z - \mathbb{P}^{(s_1)}\mathbb{P}'^{(s_2)}Z\\ &\quad= \big(\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z\big)s_1 s_2\\ &\qquad + \Big(V_1[Z;s_1] - V_1\big[\big(Z - (\nabla_Y Z)s_2 + V_2[Z;s_2]s_2^2\big); s_2\big]\Big)s_1^2\\ &\qquad + \Big(V_2\big[\big(Z - (\nabla_X Z)s_1 + V_1[Z;s_1]s_1^2\big); s_2\big] - V_2[Z;s_2]\Big)s_2^2\end{aligned}$$

$$-\Big((\nabla_Y V_1[Z;s_1]) - (\nabla_X V_2[Z;s_2])\Big)s_2^2 s_1.$$

In particular, using (5) for the second and third lines in the right hand side, we have:

$$\lim_{(s_1,s_2)\to(0,0)} \frac{\mathbb{P}'^{(s_2)}\mathbb{P}^{(s_1)}Z - \mathbb{P}^{(s_1)}\mathbb{P}'^{(s_2)}Z}{s_1 s_2} = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z = R(Y,X)Z.$$

Using the fact that $\mathbb{P}^{(-h)}\mathbb{P}^{(h)} = \mathrm{Id}$ (and similarly for $\mathbb{P}'$), we have

$$\begin{aligned} Z_{(s_1,s_2)}|_p - Z|_p &= \big(\mathbb{P}'^{(-s_2)}\mathbb{P}^{(-s_1)}\mathbb{P}'^{(s_2)}\mathbb{P}^{(s_1)}Z\big)|_{\phi(0,0)} - Z|_{\phi(0,0)} \\ &= \Big(\mathbb{P}'^{(-s_2)}\mathbb{P}^{(-s_1)}\Big[\mathbb{P}'^{(s_2)}\mathbb{P}^{(s_1)}Z - \mathbb{P}^{(s_1)}\mathbb{P}^{('s_2)}Z\Big]\Big)\Big|_{\phi(0,0)} \end{aligned}$$

Therefore, since $\lim_{h\to 0}\mathbb{P}^{(-h)} = \mathrm{Id}$ (and similarly for $\mathbb{P}'$), we obtain the required formula:

$$\lim_{(s_1,s_2)\to(0,0)} \frac{Z_{(s_1,s_2)}|_p - Z|_p}{s_1 s_2} = R(Y,X)Z|_p.$$